INTRODUCTION TO
DIFFERENTIAL EQUATIONS

Dr. Kalipada Maity

Associate Professor
Department of Mathematics
Mugberia Gangadhar Mahavidyalaya, Bhupatinagar
Purba Medinipur-721425, West Bengal, India

To my beloved Daughters

Samadrita & Somdatta

Preface

With the remarkable advancement in various branches of science, engineering and technology, today more than ever before, the study of differential equations has become essential. For, to have an exhaustive understanding of subjects like physics, mathematical biology, chemical science, mechanics, fluid dynamics, heat transfer, aerodynamics, electricity, waves and electromagnetic, the knowledge of finding solution to differential equations is absolutely necessary. These differential equations may be ordinary or partial. Finding and interpreting their solutions are at the heart of applied mathematics. A thorough introduction to differential equations is therefore a necessary part of the education of any applied mathematician, and this book is aimed at building up skills in this area.

This book on ordinary / partial differential equations is the outcome of a series of lectures delivered by me, over several years, to the undergraduate or postgraduate students of Mathematics at various institution. My principal objective of the book is to present the material in such a way that would immediately make sense to a beginning student. In this respect, the book is written to acquaint the reader in a logical order with various well-known mathematical techniques in differential equations. Besides, interesting examples solving JAM / GATE / NET / IAS / NBHM/TIFR/SSC questions are provided in almost every chapter which strongly stimulate and help the students for their preparation of those examinations from graduate level.

Organization of the book
The book has been organized in a logical order and the topics are discussed in a systematic manner. It has comprising 21 chapters altogether. In the chapter **??**, the fundamental concept of differential equations including autonomous/ non-autonomous and linear / non-linear differential equations has been explained. The order and degree of the ordinary differential equations (ODEs) and partial differential equations(PDEs) are also mentioned. The chapter **??** are concerned the first order and first degree ODEs. It is also written in a progressive manner, with the aim of developing a deeper understanding of ordinary differential equations, including conditions for the existence and uniqueness of solutions. In chapter **??** the first order and higher degree ODEs are illustrated with sufficient examples. The chapter **??** is concerned with the higher order and first degree ODEs. Several methods, like method of undetermined coefficients, variation of parameters and Cauchy-Euler equations are also introduced in this chapter. In chapter **??**, second order initial value problems, boundary value problems and Eigenvalue problems with Sturm-Liouville problems are expressed with proper examples. Simultaneous linear differential equations are studied in chapter **??**. It is also written in a progressive manner with the aim of developing some alternative methods. In chapter **??**, the equilibria, stability

and phase plots of linear / nonlinear differential equations are also illustrated by including numerical solutions and graphs produced using Mathematica version 9 in a progressive manner. The geometric and physical application of ODEs are illustrated in chapter **??**. The chapter 1 is presented the Total (Pfaffian) Differential Equations. In chapter **??**, numerical solutions of differential equations are added with proper examples. Further, I discuss Fourier transform in chapter **??**, Laplace transformation in chapter **??**, Inverse Laplace transformation in chapter **??**. Moreover, series solution techniques of ODEs are presented with Frobenius method in chapter **??**, Legendre function and Rodrigue formula in Chapter **??**, Chebyshev functions in chapter **??**, Bessel functions in chapter **??** and more special functions for Hypergeometric, Hermite and Laguerre in chapter **??** in detail.

Besides, the partial differential equations are presented in chapter **??**. In the said chapter, the classification of linear, second order partial differential equations emphasizing the reasons why the canonical examples of elliptic, parabolic and hyperbolic equations, namely Laplace's equation, the diffusion equation and the wave equation have the properties that they do has been discussed. Chapter **??** is concerned with Green's function. In chapter **??**, the application of differential equations are developed in a progressive manner. Also all chapters are concerned with sufficient examples. In addition, there is also a set of exercises at the end of each chapter to reinforce the skills of the students.

Moreover it gives the author great pleasure to inform the reader that the **second edition** of the book has been improved, well -organized, enlarged and made up-to-date as per latest UGC - CBSC syllabus. The following significant changes have been made in the second edition:

- Almost all the chapters have been rewritten in such a way that the reader will not find any difficulty in understanding the subject matter.
- Errors , omissions and logical mistakes of the previous edition have been corrected.
- The exercises of all chapters of the previous edition have been improved, enlarged and well-organized.
- Two new chapters like Green's Functions and Application of Differential Equations have been added in the present edition.
- More solved examples have been added so that the reader may gain confidence in the techniques of solving problems.
- References to the latest papers of various university, IIT-JAM, GATE, and CSIR-UGC(NET) have been provided in almost every chapters which strongly help the students for their preparation of those examinations from graduate label.

In view of the above mentioned features it is expected that this new edition will appreciate and be well prepared to use the wonderful subject of differential equations.

Aim and Scope
When mathematical modelling is used to describe physical, biological or chemical phenomena, one of the most common results of the modelling process is a system of ordinary or partial differential equations. Finding and interpreting the solutions of these differential equations

is therefore a central part of applied mathematics, Physics and a thorough understanding of differential equations is essential for any applied mathematician and physicist. The aim of this book is to develop the required skills on the part of the reader. The book will thus appeal to undergraduates/postgraduates in Mathematics, but would also be of use to physicists and engineers. There are many worked examples based on interesting real-world problems. A large selection of examples / exercises including JAM/NET/GATE questions is provided to strongly stimulate and help the students for their preparation of those examinations from graduate level. The coverage is broad, ranging from basic ODE , PDE to second order ODE's including Bifurcation theory, Sturm-Liouville theory, Fourier Transformation, Laplace Transformation, Green's function and existence and uniqueness theory, through to techniques for nonlinear differential equations including stability methods. Therefore, it may be used in research organization or scientific lab.

Significant features of the book

- A complete course of differential Equations
- Perfect for self-study and class room
- Useful for beginners as well as experts
- More than 650 worked out examples
- Large number of exercises
- More than 700 multiple choice questions with answers
- Suitable for New UGC-CBSC syllabus of ODE & PDE
- Suitable for GATE, NET, NBHM, TIFR, JAM, JEST, IAS, SSC examinations.

ACKNOWLEDGEMENTS

This book is the outcome of a series of lectures and research experience carried out by me over several years. However it would not be possible to incorporate or framing the entire book without the help of many academicians. As such, I am indebted to many of my teachers and students. Especially I would like to thank Dr. Swapan Kumar Misra, Principal, Mugberia Gangadhar Mahavidyalaya for his generous support in writing this book.

I express sincerest respect to my research guide Prof. M. Maiti, Department of Mathematics, Vidyasagar University, Midnapore 721102, West Bengal, India.

I also express sincerest gratitude to my collogues: Dr. Nabakumar Ghosh, Dr. Arpan Dhara, Prof. Bikash Panda, Prof. Hiranmay Manna, Prof. Madhumita Sahoo for their positive suggestions to improve the standard of the book. Especial thanks are owed to my collogues, Dept. of Mathematics: Prof. Suman Giri, Prof. Asim Jana, Prof. Tanushri Maity, Prof. Debraj Manna for their excellent typing to add some chapters of the book. Also I wish to thank several persons of our institution who have made many encouraging remarks and constructive suggestions on the manuscript.

I would like to express sincere appreciation to my friends: Dr. Dipak Kr. Jana and Dr. Shibsankar Das for their constant source of inspiration.

My sincere appreciation goes to my students who give me a stage where I can cultivate my talent and passion for teaching. My graduate and post graduate students who have used the draft of this book as a textbook have made many encouraging comments and constructive suggestions. Also, I heartily thanks my scholars: Dr. Samar Hazari, Mr. Jatin Nath Roul, Mr. Anupam De and Mr. Debnarayan Khatua for their help in different direction to modify the book.

Without the unfailing love and support of my parents, who have always believed in me, this work would not have been possible. In addition, the care, love, patience, and understanding of my wife and lovely daughters have been of inestimable encouragement and help. I love them very much and appreciate all that they have contributed to my work.

In written this book, I have taken some helps from several books and research papers which are immensely helpful and are given in an alphabetical order in the bibliography. So I also express my gratitude and respect to all the eminent authors. I apologize if I inadvertently missed acknowledging mygratitude to any one else.

I shall feel great to receive constructive criticisms through email for the improvement of the book from the experts as well as the learners.

I thank the Narosa Publishing House Pvt. Ltd. for their sincere care in the publication of this book.

Kalipada Maity,
E-mail: kalipada_maity@yahoo.co.in

Contents

Chapter 1

Total (Or Pfaffian) Differential Equations

1.1 Introduction

In this chapter, we proposed to discuss differential equations with one independent variable and more than one dependent variables.

Definition 1.1 (**Pfaffian Differential Equation:**) Let $u_i,\ i = 1, 2, \cdots, n$ be n functions of some or all of n independent variables $x_1, x_2, \cdots, x_n$. Then $\sum_{i=1}^{n} u_i dx_i$ is called a Pfaffian differential form in n variables and $\sum_{i=1}^{n} u_i dx_i = 0$ is called a Pfaffian differential equation in n variables $x_1, x_2, \cdots, x_n$.

Definition 1.2 (**Total(Pfaffian)Differential Equation for Three Variables:**)

An equation of the form $$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0 \tag{1.1}$$

is called the Pfaffian Differential Equation in three variables x, y, z.

The equation (1.1) can be directly integrated if there exists a function $u(x, y, z)$ whose total differential du is equal to the left hand number of (1.1). In other cases (1.1) may or may not be integrable. We now proceed to find the condition which P, Q, R must satisfy, so that (1.1) may be integrable. This will be called the condition or criteria of integrability of this single differential equation (1.1).

1.2 Necessary and sufficient conditions for integrability of total(or single) differential equation $Pdx + Qdy + Rdz = 0$.

1.2.1 Necessary condition:

Consider the total (or single) differential equation

$$Pdx + Qdy + Rdz = 0 \quad \text{where } P, Q, R \text{ are functions of } x, y, z. \tag{1.2}$$

$$\text{Let (1.2) have an integral} \quad u(x, y, z) = c \tag{1.3}$$

Then total differential du must be equal to $Pdx + Qdy + Rdz$, or to it multiplied by a factor. But, we know that

$$du = (\frac{\partial u}{\partial x})dx + (\frac{\partial u}{\partial y})dy + (\frac{\partial u}{\partial z})dz. \tag{1.4}$$

Since (1.3) is an integral of (1.2), P, Q, R must be proportional to $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$. Therefore, $\frac{\frac{\partial u}{\partial x}}{P} = \frac{\frac{\partial u}{\partial y}}{Q} = \frac{\frac{\partial u}{\partial z}}{R} = \lambda(x, y, z)$ (say).

$$\lambda P = \frac{\partial u}{\partial x}, \qquad \lambda Q = \frac{\partial u}{\partial y}, \qquad \text{and } \lambda R = \frac{\partial u}{\partial z} \tag{1.5}$$

From the first two equations of (1.5), we get

$$\frac{\partial}{\partial y}(\lambda P) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}\Big) = \frac{\partial}{\partial x}(\lambda Q)$$

$$\text{or, } \lambda\frac{\partial P}{\partial y} + P\frac{\partial \lambda}{\partial y} = \lambda\frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x}$$

$$\text{or, } \lambda\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) = Q\frac{\partial \lambda}{\partial x} - P\frac{\partial \lambda}{\partial y} \tag{1.6}$$

$$\text{Similarly} \qquad \lambda\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) = R\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial z} \tag{1.7}$$

$$\text{and} \qquad \lambda\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) = P\frac{\partial \lambda}{\partial z} - R\frac{\partial \lambda}{\partial x} \tag{1.8}$$

Multiplying (1.6)-(1.8) by R, P and Q respectively and adding, we get

$$P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) = 0. \tag{1.9}$$

This is therefore, the necessary condition for the integrability of the equation (1.2).

1.2.2 Sufficient condition:

Suppose that the coefficients P, Q, R of (1.2) satisfy the relation (1.9). It will now be proved that this relation gives the required sufficient condition for the existence of an integral of (1.2). For this we show that an integral of (1.2) can be found when relation (1.9) holds. We first prove that if we take $P_1 = \mu P$, $Q_1 = \mu Q$, $R_1 = \mu R$ where μ is any function of x, y and z, the same condition is satisfied by P_1, Q_1, R_1 as by P, Q, R. We have

$$\frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y} = \mu\frac{\partial Q}{\partial z} + Q\frac{\partial \mu}{\partial z} - \Big(\mu\frac{\partial R}{\partial y} + R\frac{\partial \mu}{\partial y}\Big)$$

$$\text{or, } \frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y} = \mu\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial y} \tag{1.10}$$

$$\text{Similarly, } \frac{\partial R_1}{\partial x} - \frac{\partial P_1}{\partial z} = \mu\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial z} \tag{1.11}$$

$$\text{and } \frac{\partial P_1}{\partial y} - \frac{\partial Q_1}{\partial x} = \mu\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) + P\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial x} \tag{1.12}$$

Multiplying (1.10), (1.11) and (1.12) by P_1, Q_1, R_1 respectively, adding and replacing P_1, Q_1, R_1 by μP, μQ, μR respectively in resulting R.H.S., we obtain

$$P_1\Big(\frac{\partial Q_1}{\partial z}-\frac{\partial R_1}{\partial y}\Big)+Q_1\Big(\frac{\partial R_1}{\partial x}-\frac{\partial P_1}{\partial z}\Big)+R_1\Big(\frac{\partial P_1}{\partial y}-\frac{\partial Q_1}{\partial x}\Big) \tag{1.13}$$

$$=\mu\Big\{P\Big(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\Big)+Q\Big(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\Big)+R\Big(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\Big)\Big\}=0 \text{ using (1.9)} \tag{1.14}$$

Now $Pdx+Qdy$ may be regarded as an exact differential. For if it is not so, then multiplying the equation (1.2) by the integrating factor $\mu(x,y,z)$, we can make it so. Thus there is no loss of generality in regarding $Pdx+Qdy$ as an exact differential. For this the condition is

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x} \tag{1.15}$$

$$\text{Let } V=\int(Pdx+Qdy) \tag{1.16}$$

$$\text{then it follows that } P=\frac{\partial V}{\partial x} \qquad \text{and } Q=\frac{\partial V}{\partial y} \tag{1.17}$$

$$\text{From (1.17), } \frac{\partial P}{\partial z}=\frac{\partial^2 V}{\partial z\partial x} \qquad \textit{and} \qquad \frac{\partial Q}{\partial z}=\frac{\partial^2 V}{\partial z\partial y}. \tag{1.18}$$

By using the above relation, (1.15), (1.17) and (1.9) gives

$$\frac{\partial V}{\partial x}\Big(\frac{\partial^2 V}{\partial z\partial x}-\frac{\partial R}{\partial y}\Big)+\frac{\partial V}{\partial y}\Big(\frac{\partial R}{\partial x}-\frac{\partial^2 V}{\partial z\partial x}\Big)=0 \qquad \text{or} \qquad \frac{\partial V}{\partial x}\frac{\partial}{\partial y}\Big(\frac{\partial V}{\partial z}-R\Big)-\frac{\partial V}{\partial y}\frac{\partial}{\partial x}\Big(\frac{\partial V}{\partial z}-R\Big)=0$$

$$\text{or} \quad \begin{vmatrix} \frac{\partial V}{\partial x} & \frac{\partial}{\partial x}\Big(\frac{\partial V}{\partial z}-R\Big) \\ \frac{\partial V}{\partial y} & \frac{\partial}{\partial y}\Big(\frac{\partial V}{\partial z}-R\Big) \end{vmatrix}=0.$$

This show that a relation independent of x and y exists between V and $(\frac{\partial V}{\partial z})-R$. Consequently $(\frac{\partial V}{\partial z})-R$ can be expressed as a function of z and V alone. That is we can take

$$\frac{\partial V}{\partial z}-R=\phi(z,V). \tag{1.19}$$

$$\text{Now, } Pdx+Qdy+Rdz \;=\; \frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\Big(\frac{\partial V}{\partial z}-\phi\Big)dz, \text{ using (1.16) and (1.19)}$$

$$\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\frac{\partial V}{\partial z}dz-\phi dz=dV-\phi dz. \tag{1.20}$$

Thus (1.2) may be written as $dV-\phi dz=0$ which is an equation in two variables. Hence its integration will gives an integral of the form. Hence the condition (1.9) is sufficient. Thus (1.9) is both the necessary and sufficient condition that (1.2) has an integral.

Theorem 1.1 Prove that the necessary and sufficient condition for integrability of the total differential equation $\mathbf{A}.d\mathbf{r}=Pdx+Qdy+Rdz=0$ is $\mathbf{A}.\mathbf{curl}\,\mathbf{A}=0$.

Proof.

$$\text{Given } \mathbf{A}.d\mathbf{r}=Pdx+Qdy+Rdz=0. \tag{1.21}$$

$$\text{Let } \mathbf{r}=x\hat{i}+y\hat{j}+z\hat{k} \qquad \text{so that} \qquad d\mathbf{r}=dx\hat{i}+dy\hat{j}+dz\hat{k} \tag{1.22}$$

$$\text{and} \qquad \mathbf{A}=P\hat{i}+Q\hat{j}+R\hat{k} \tag{1.23}$$

Then we see that (1.21) is satisfied by usual rule of dot product of two vectors **A** and $d\mathbf{r}$. No show that the necessary condition for integrability of (1.21) is

$$P\Big(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\Big)+Q\Big(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\Big)+R\Big(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\Big)=0. \tag{1.24}$$

From vector calculus, we have

$$\textbf{Curl A}=\Big(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\Big)\hat{i}+\Big(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\Big)\hat{j}+\Big(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\Big)\hat{k} \tag{1.25}$$

Hence, using (1.23) and (1.25) and applying the usual rule of dot product of two vectors, the necessary condition (1.24) may be rewritten as **A.curl A** $= 0$ as desired.

1.3 The conditions for exactness of $Pdx + Qdy + Rdz = 0$.

The given total differential equation is said to be exact if the following three conditions are satisfied.

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x},\ \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}\ \text{and}\ \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}. \tag{1.26}$$

Note that when condition (1.26) are satisfied, the condition for integrability of $Pdx+Qdy+Rdz=0$, namely,

$$P\Big(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\Big)+Q\Big(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\Big)+R\Big(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\Big)=0 \tag{1.27}$$

is also satisfied for each term of (1.27), vanishes identically.

1.4 Show that the locus of $Pdx + Qdy + Rdz = 0$ is orthogonal to the locus of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\text{The equation}\ \ Pdx + Qdy + Rdz = 0 \tag{1.28}$$

means, geometrically that a straight line whose direction cosines are proportional to dx, dy, dz is perpendicular to a line whose direction cosines are proportional to P, Q, R. As a consequence a point which satisfies (1.28) must move in a direction at right angles to a line whose direction cosines are proportional to P, Q, R. On the other hand, the equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \tag{1.29}$$

mean geometrically that a straight line whose direction cosines are proportional to dx, dy and dz. Again from the equation (1.29) (i.e. $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$), we see that dx, dy and dz are proportional to P, Q and R. Thus (P, Q, R) are the corresponding direction rations of the tangents to the curves

at (x, y, z). Thus geometrically the above differential equations represents a system of curves in space such that the direction cosines of the tangent to these curves at any point (x, y, z) are proportional to (P, Q, R).
From the above discussion it follows that the curves traced out by the points that are moving according to the condition (1.28) are orthogonal to the curves traced out by the points that are moving according to the conditions (1.28). The former curves are any of the curves upon the surfaces given by (1.28). Thus geometrically, the curves represented by (1.29) are normal to the surfaces represented by (1.28). In case (1.28) is not integrable, there can not exist a family of surfaces which is orthogonal to all lines that form the locus of (1.29).

1.5 Geometrical Interpretation of $Pdx + Qdy + Rdz = 0$

The given differential equation expresses that the tangent to a curve is perpendicular to a certain line, the direction cosines of this tangent line and another line being proportional to dx, dy, dz and P, Q, R respectively. Suppose that the equation

$$Pdx + Qdy + Rdz = 0 \tag{1.30}$$

satisfies the condition of integrability and that its solution is

$$F(x, y, z, c) = 0. \tag{1.31}$$

Since (1.31) has one arbitrary constant, it represents a single infinity of surfaces. Choosing this constant in an appropriate manner, (1.31) can be made to pass through any given point of space. If a point is moving upon this surface in any direction, its co-ordinates and direction cosines of its path at any moment must satisfy (1.30), since (1.31) is the integral of (1.30). Again for each point (x, y, z) there will be an infinite number of values of dx, dy, dz which will satisfy (1.30). Thus it follows that a point which is moving in such a manner that its co-ordinates and the direction cosines of its path always satisfy (1.30) can pass through any point in an infinity of directions. However, while passing through any point, it must remain on the particular surface given by (1.31) which passes through the point. Thus, infinite number of such possible curves which it can describe through that point must lie on the surface.

1.6 Methods of solution of the Total Differential Equation

In this section, we discuss various type of methods from which a suitable one will be used to obtain the corresponding integral of the total differential equation. At first the conditions of integrability as given by (1.9) should be verified first, then the suitable method for the determination of the corresponding integral as given below will be considered.

1.6.1 Method I(Solution by inspection)

By rearranging the terms of the given equation or by dividing by a suitable functions of x, y, z to reduce some part of the equation into exact differentials and then integrating, the required integral is determined.

Example 1.1 Solve $(yz+2x)dx+(zx-2z)dy+(xy-2y)dz=0$.

Solution: Comparing the given equation with $Pdx+Qdy+Rdz=0$, we get

$$
\begin{aligned}
&P=yz+2x\,, \qquad Q=zx-2z\,, \qquad R=xy-2y.\\
&\therefore P\Big(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\Big)+Q\Big(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\Big)+R\Big(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\Big)\\
&=(yz+2x)\{(x-2)-(x-2)\}+(zx-2z)(y-y)+(xy-2y)(z-z)=0
\end{aligned}
$$

Showing that the given total differential equation is integrable. On rearranging, the given equation can be written as

$$
\begin{aligned}
&(yzdx+zxdy+xydz)+2xdx-2(zdy+ydz)=0\\
&\Rightarrow d(xyz)+d(x^2)-2d(yz)=0\\
&\text{Integrating we get}\,,\ xyz+x^2-2yz=c,
\end{aligned}
$$

which is the required general solution, c being an arbitrary constant.

1.6.2 Method II(Solution of homogeneous equation)

The equation $Pdx+Qdy+Rdz=0$ is called a homogeneous equation if P, Q, R are homogeneous functions of x, y, z of the same degree.

Step I: As usual verify that the given equation is integrable.

Step II: Put $x=zu$, $y=zv$ so that $dx=udz+zdu$ and $dy=zdv+vdz$. Substituting these in the given equation, we solve the equation.

Example 1.2 Solve $2(y+z)dx-(x+z)dy+(2y-x+z)dz=0$

Solution: Comparing the given equation with $Pdx+Qdy+Rdz=0$, we get, $P=2y+2z$, $Q=-x-z$, $R=2y-x+z$ and $\sum P(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y})=0$. So the condition of integrability is satisfied.
The given equation is homogeneous. Let us consider $x=uz$, $y=vz$ so as to obtain $dx=udz+zdu$, $dy=vdz+zdv$. Then the given equation becomes

$$
\begin{aligned}
&z[2(v+1)du-(u+1)dv]+(u+1)(v+1)dz=0\\
&\text{or,}\ \ 2\frac{du}{u+1}-\frac{dv}{v+1}+\frac{dz}{z}=0
\end{aligned}
$$

Integrating we get, $(u+1)^2z=c(v+1)$ or $(x+z)^2=c(y+z)$ which is the required general solution and c being an arbitrary constant.

1.6.3 Method III(Use of auxiliary equation)

Let the total differential equation $Pdx+Qdy+Rdz=0$ be integrable. Then it follows that P, Q, R satisfy

$$P\Big(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\Big)+Q\Big(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\Big)+R\Big(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\Big)=0$$

Now comparing these two equations we have

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}} \tag{1.32}$$

known as the auxiliary equations. Then the above auxiliary equations is solved using the methods (for solving simultaneous equation of type-II) as described in the chapter-**??**.
Let $u = c_1$, $v = c_2$ (where c_1, c_2 being arbitrary constants) be the integrals obtained by solving equation (1.32). Now we find two functions ϕ and ψ such that the equation may be written by $\phi du + \psi dv = 0$.

In fact as u and v are known, the equation $\phi du + \psi dv = 0$ will reduce to an equation of the form $P_1 dx + Q_1 dy + R_1 dz = 0$ and comparing this with the original equation the values of the function ϕ and ψ are determined and finally with these values of ϕ and ψ, the equation $\phi du + \psi dv = 0$ can be integrated to obtain the required integral of the given equation.

Example 1.3 Solve $xz^3 dx - zdy + 2ydz = 0$.

Solution: Here,

$$xz^3 dx - zdy + 2ydz = 0 \tag{1.33}$$

Comparing (1.33) with $Pdx + Qdy + Rdz = 0$, here

$$P = xz^3, \;\; Q = -z, \;\; R = 2y \tag{1.34}$$

Here $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = xz^3 \cdot (-1-2) - z \cdot (0 - 3xz^2) + 2y \cdot 0 = -3xz^3 + 3xz^3 = 0$.
Thus the condition of integrability is satisfied.
The auxiliary equations of the given equation are

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}} \tag{1.35}$$

$$\frac{dx}{-1-2} = \frac{dy}{0 - 3xz^2} = \frac{dz}{0} \tag{1.36}$$

$$\frac{dx}{1} = \frac{dy}{xz^2} = \frac{dz}{0} \tag{1.37}$$

Taking the third ratio of (1.37), we have

$$dz = 0 \qquad \text{so that } z = c_1 = u \text{ (say)} \tag{1.38}$$

Taking the first and second ratios of (1.37), we have

$$xz^2 dx - dy = 0 \tag{1.39}$$

Integrating,

$$x^2u^2 - 2y = c_2 = v \text{ (say)} \tag{1.40}$$

$$x^2z^2 - 2y = v \text{ using (1.38)} \tag{1.41}$$

Substituting the value of u and v from (1.37) and (1.38) in $Adu + Bdv = 0$ we get $Adz + B(2xz^2dx + 2x^2zdz - 2dy) = 0$

$$2Bxz^2dx - 2Bdy + (A + 2Bx^2z)dz = 0 \tag{1.42}$$

Comparing (1.33) with (1.42), we get $xz^3 = 2Bxz^2,\; 2y = A + 2Bx^2z$ i.e., $B = \frac{u}{2},\; A = -v$. Substituting these values of A and B in $Adu + Bdv = 0$, we get $-vdu + \frac{udv}{2} = 0 \Rightarrow \frac{dv}{v} = \frac{2du}{u}$. Integrating we get, $\log v = 2\log u + \log c \Rightarrow v = cu^2$ Using (1.38) and (1.41), we get $x^2z^2 - 2y = cz^2$ which is required general solution.

1.6.4 Method IV(General method of solving $Pdx + Qdy + Rdz = 0$ by taking one variable as constant)

Step 1. First verify the integrability condition
Step 2. We now treat one of the variables, say z as a constant i.e., $dz = 0$, then the resulting equation is reduced to

$$Pdx + Qdy = 0 \text{ using} \tag{1.43}$$

We should select a proper variable to be constant so that the resulting equation in the remaining variables is easily integrable. Thus this selection will vary from problem to problem. The present discussion is for the choice z = constant. For other cases the necessary changes have to be made in the entire procedure.
Step 3. Let the solution of (1.43) by $u(x, y, z) = f(z)$, where $f(z)$ is an arbitrary function of z ($\because z$ = constant). Thus the solution of (1.43) is $u(x, y) = f(z)$.
Step 4. We now differentiate $u(x, y) = f(z)$ *w.r.t.* x, y, z and then compare the result with the given equation $Pdx + Qdy + Rdz = 0$. After comparing we shall get an equation in two variables f and z. If the coefficient of f or z involve functions of x, y, it will always be possible to remove them with the help of $u(x, y) = f(z)$.
Step 5. Solve the equation got in step 4 and obtain f. Putting this value of f in $u(x, y) = f(z)$, we shall get the required solution of the required equation.

Example 1.4 Solve $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$

Solution: Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$\begin{aligned}
&P = 3x^2, \qquad Q = 3y^2, \qquad R = -(x^3 + y^3 + e^{2z}).\\
&\therefore P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big)\\
&= 0
\end{aligned}$$

Showing that the condition of integrability is satisfied. Let z be treated as constant, so that $dz = 0$. Then the given equation becomes $3x^2dx + 3y^2dy = 0$. Integrating we get

$$x^3 + y^3 = f(z) \text{ (say) } (\because\ z \text{ is constant}) \tag{1.44}$$

Differentiating (1.44), we get

$$3x^2dx + 3y^2dy - f'(z)dz = 0 \tag{1.45}$$

Comparing (1.45) with the given equation $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$ we have, $f'(z) = x^3 + y^3 + e^{2z} \Rightarrow f'(z) = f(z) + e^{2z}$. Hence its general solution is $f(z) = e^{2z} + ce^{z} \Rightarrow x^3 + y^3 = e^{2z} + ce^{z}$, where c is the required solution.

1.6.5 Method V(Exact and homogeneous of degree $n \neq -1$)

Theorem 1.2 If the total differential equation $Pdx + Qdy + Rdz = 0$ is exact and homogeneous of degree $n \neq -1$, then its general solution is given by $Px + Qy + Rz = c$ where c is an arbitrary constant.

Solution: The given total differential equation is $Pdx + Qdy + Rdz = 0$ and so we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \ \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}. \tag{1.46}$$

Also $Pdx + Qdy + Rdz = 0$ is homogeneous of degree $n \neq -1$, so by Euler's theorem on homogeneous functions of x, y, z of degree $n \neq -1$, we have

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} = nP \tag{1.47}$$

$$x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z} = nQ \tag{1.48}$$

$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z} = nR \tag{1.49}$$

Now adding $Pdx + Qdy + Rdz = 0$ with the equations (1.47) - (1.49), we get

$$\begin{aligned} & x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z} + x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z} \\ + \quad & (Pdx + Qdy + Rdz) = (n+1)(Pdx + Qdy + Rdz) = 0 \end{aligned} \tag{1.50}$$

Using the equation (1.46), from (1.50) we get

$$\begin{aligned} & \left(P + x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z}\right)dx + \left(Q + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z}\right)dy \\ & \left(R + x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z}\right)dz = 0 \end{aligned} \tag{1.51}$$

$$d(Px + Qy + Rz) = 0 \tag{1.52}$$

Integrating we get, $Px + Qy + Rz = c$ is the solution of the total differential equation $Pdx + Qdy + Rdz = 0$ which is exact and homogeneous in x, y, z of degree $n \neq -1$. Hence the theorem.

Example 1.5 Show that $(y+z)dx + (z+x)dy + (x+y)dz = 0$ is exact and homogeneous. Hence solve it.

Solution: Comparing the given total differential equation with $Pdx + Qdy + Rdz = 0$, we get $P = y + z,\ Q = z + x,\ R = x + y$. Then

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 1 - 1 = 0 \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} = 1 - 1 = 0 \Rightarrow \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

$$\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = 1 - 1 = 0 \Rightarrow \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

Hence the given total differential equation is exact. Also $P = y(1 + \frac{z}{y}),\ Q = z(1 + \frac{x}{z}),\ R = x(1 + \frac{y}{x})$. So $P,\ Q,\ R$ are homogeneous functions of degree 1. Hence the required solution is given by $Px + Qy + Rz = c$ where c being an arbitrary constant. Hence $2xy + 2yz + 2zx = c$ is the required solution.

1.7 Non-integrable Single Differential Equation

Let us consider the single differential equation

$$Pdx + Qdy + Rdz = 0 \tag{1.53}$$

where P, Q, R are functions of x, y, z. If the equation (1.53) does not satisfy the condition of integrability, then there exists no singular relation between x, y, z to satisfy it. However the equation (1.53) represents a family of curves orthogonal to the family represented by $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Thus there exists an infinite number of curves that lie on any given surface and satisfy (1.53). Consequently the method of finding the above mentioned infinite number of curves is equally applicable to integrable equation (1.53).

Let the curves represented by the solution of the non-integrable equation (1.53) lie on the surface represented by

$$u(x, y, z) = c \tag{1.54}$$

where c being an arbitrary constant. Differentiating (1.54) we have

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \tag{1.55}$$

From (1.55) by eliminating the variables z and dz by using (1.53) and (1.54) we get an equation of the form $Mdx + Ndy = 0$, where M and N are functions of x, y. Solving this we get relation between x, y involving an arbitrary constant and this relation with equation (1.55) give the desire curves i.e., the solution of (1.53).

Example 1.6 Show that there is no single integral of $dz = 2ydx + xdy$. Prove that the curve of the equation that lie in the plane $z = x + y$ lie also on surface of the family $(x-1)^2(2y-1) = c$.

Solution: Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$\begin{aligned}
&P = 2y, \qquad Q = x, \qquad R = -1. \\
&\therefore P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) \\
&= 2y \cdot 0 + x \cdot 0 - 1(2-1). \\
&= -1 \neq 0
\end{aligned}$$

Showing that the condition of integrability is not satisfied.

Also from $z = x + y$, we get, $dz = dx + dy$. Then using $2ydx + xdy - dz = 0$ and $dz = dx + dy$, we have,

$$\begin{aligned}
&(2y-1)dx + (x-1)dy = 0 \\
&\frac{2}{x-1}dx + \frac{2}{2y-1}dy = 0
\end{aligned}$$

Integrating we get, $2\log(x-1) + \log(2y-1) = \log c$ i.e., $(x-1)^2(2y-1) = c$.

1.8 Worked Out Example

Example 1.7 Show that $(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)dz = 0$ is integrable. Hence solve it.
Solution: Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$\begin{aligned}
&P = yz + xyz\,, \qquad Q = zx + xyz\,, \qquad R = xy + xyz.\\
&\therefore P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big)\\
&= yz(1 + x)\{(x + xy) - (x + xz)\} + zx(1 + y)\{(y + yz) - (y + xy)\} + xy(1 + z)\{(z + xz) - (z + yz)\}\\
&= yz(1 + x)x(y - z) + zx(1 + y)y(z - x) + xy(1 + z)z(x - y)\\
&= xyz\{(1 + x)(y - z) + (1 + y)(z - x) + (1 + z)z(x - y)\}\\
&= xyz[0 - 0] = 0.
\end{aligned}$$

Showing that the condition of integrability is satisfied.
Dividing each term by xyz, the given equation becomes

$$(\frac{1}{x} + 1)dx + (\frac{1}{y} + 1)dy + (\frac{1}{z} + 1)dz = 0$$

Integrating, we get, $\log x + x + \log y + y + \log z + z = c$ or $\log(xyz) + x + y + z = c$, which is the required general solution, where c being an arbitrary constant.

Example 1.8 Show that $z(z - y)dx + z(z + x)dy + x(x + y)dz = 0$ is an integrable. Hence solve it.
Solution: Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$\begin{aligned}
&P = z(z - y)\,, \qquad Q = z(z + x)\,, \qquad R = x(x + y).\\
&\therefore P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big)\\
&= z(z - y)(2z + x - x) + z(z + x)(2x + y - 2z + y) + x(x + y)(-z - z) = 0,
\end{aligned}$$

which show that the condition of integrability is satisfied. So it is an integrable.
The auxiliary equations of the given equation are

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dy}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}} \tag{1.56}$$

$$\Rightarrow \quad \frac{dx}{z} = \frac{dy}{x + y - z} = \frac{dz}{-z}. \tag{1.57}$$

From first and third ratios, we get $dx + dz = 0$. Integrating, $x + z = c_1$(constant) or $u = c_1$ where $u = x + z$. Again from (1.57), we have $\frac{dx+dy}{x+y} = \frac{dz}{-z}$. Integrating, $\log(x + y) + \log z = \log c_2 \Rightarrow (x + y)z = c_2$ or $v = c_2$ where $v = (x + y)z$.
Now we wish to find ϕ and ψ s.t the given equation is identical with

$$\phi du + \psi dv = 0 \tag{1.58}$$

Then using $u = c_1$, $v = c_2$ we have from (1.58),

$$\begin{aligned}
&\phi d(x + y) + \psi d(xz + yz) = 0\\
\Rightarrow\ &(\phi + z\psi)dx + z\psi dy + (\phi + x + y)dz = 0
\end{aligned}$$

Comparing this equation with the given equation, we get $\phi + z\psi = kz(z-y)$, $z\psi = kz(z+x)$ and $\phi + x + y = kx(x+y)$, where k is non zero constant. Now solving the above relations, we get $\psi = k(x+z) = ku$ and $\phi = kz(z-y) - z\psi = kz(z-y) - kz(z+x) = -ku$. Using the values of ϕ and ψ in (1.58) becomes $-vdu + udv = 0 \Rightarrow \frac{du}{u} - \frac{dv}{v} = 0$. Then integrating we get, $\log u - \log v = \log c \Rightarrow u = cv$ or $x + z = cz(x+y)$ where c being an arbitrary constant and this is the required solution of the given equation.

Example 1.9 Find the orthogonal trajectories on the cone $x^2 + y^2 = z^2 \tan^2 \alpha$ of its intersection with the family of planes parallel to $z = 0$.

Solution: Given surface is

$$f(x, y, z) = x^2 + y^2 - z^2 \tan^2 \alpha = 0 \tag{1.59}$$

and the family of planes parallel to $z = 0$ is

$$z = k \tag{1.60}$$

where k is parameter. Then the system of differential equations of the given curves of intersection of (1.59) and (1.60) is given by

$$2xdx + 2ydy - 2z \tan^2 \alpha dz = 0, \qquad dz = 0 \tag{1.61}$$

Solving these equation for dx, dy, dz, we get

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0}.$$

Hence the system of differential equations of the required orthogonal trajectories of the given curves is

$$xdx + ydy - z \tan^2 \alpha dz = 0, \qquad ydx - xdy + 0 \cdot dz = 0 \tag{1.62}$$

Solving these equation for dx, dy, dz, we get

$$\frac{dx}{xz \tan^2 \alpha} = \frac{dy}{yz \tan^2 \alpha} = \frac{dz}{x^2 + y^2}. \tag{1.63}$$

Taking $x, y, 0$ as multipliers, each fraction of (1.63) $= \frac{xdx+ydy}{(x^2+y^2)z \tan^2 \alpha}$. Combining this fraction with last fraction in (1.63), we get

$$\frac{xdx + ydy}{(x^2 + y^2)z \tan^2 \alpha} = \frac{dz}{x^2 + y^2}$$

so that

$$2xdx + 2ydy - 2z \tan^2 \alpha dz = 0.$$

Integrating,

$$x^2 + y^2 - z^2 \tan^2 \alpha = k$$

where k being an arbitrary constant. Choosing $k = 0$, we obtain the given surface (1.59). Taking the first and second fractions of (1.62), we get $\frac{dx}{x} - \frac{dy}{y} = 0$. Integrating, we get, $\log x - \log y = \log c$ or $x = cy$, c being an arbitrary constant.

Hence the required family of the orthogonal trajectories is given by $x^2 + y^2 = z^2 \tan^2 \alpha$ and $x = cy$.

Example 1.10 Find the orthogonal projection of the curves on the $xy-$plane which lie on the paraboloid $3z = x^2 + y^2$ and satisfy the differential equation $2dz = (x + y)dx + ydy$.

Solution: Given paraboloid is

$$f(x, y, z) = x^2 + y^2 - 3z = 0 \tag{1.64}$$

The given differential equation is

$$(x + z)dx + ydy - 2dz = 0 \tag{1.65}$$

Comparing (1.65) with $Pdx + Qdy + Rdz = 0$, we have $P = x + y$, $Q = y$ and $R = -2$. Now

$$\begin{aligned} & P\Big(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\Big) + Q\Big(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\Big) + R\Big(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\Big) \\ &= (x + y)(0 - 0) + y(0 - 1) - 2(0 - 0) = -y \neq 0. \end{aligned}$$

Thus (1.65) does not satisfy the integrability condition. Now differentiability (1.64), we have

$$2xdx + 2ydy - 3dz = 0 \tag{1.66}$$

Then from (1.65) and (1.66), we get

$$\begin{aligned} & 3(x + z)dx + 3ydy - 2(2xdx + 2ydy) = 0 \\ \Rightarrow\ & xdx + ydy - 3zdx = 0 \\ \Rightarrow\ & xdx + ydy = (x^2 + y^2)dx, \text{ using (1.64)} \\ \Rightarrow\ & \frac{2xdx + 2ydy}{x^2 + y^2} = 2dx \end{aligned}$$

Integrating we get, $\log(x^2 + y^2) = \log c + 2x \Rightarrow x^2 + y^2 = ce^{2x} \Rightarrow 3z = ce^{2x}$ (using (1.64)) where c being an arbitrary constant.

1.9 Review Exercises

1 Show that $(2x + y^2 + 2xz)dx + 2xydy + x^2dz = 0$ is integrable.

2 Solve the following equations using Method I

(a) $yz \log zdx - zx \log zdy + xydz = 0$, **Ans.** $x \log z = cy$

(b) $(x^2z - y^3)dx + 3xy^2dy + x^3dz = 0$, **Ans.** $x^2z + y^3 = cx$

(c) $\frac{y+z-2x}{(y-x)(z-x)}dx + \frac{z+x-2y}{(z-y)(x-y)}dy + \frac{x+y-2z}{(x-z)(y-z)}dz = 0$, **Ans.** $(x - y)(y - z)(z - x) = c$

(d) $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0$, **Ans.** $xy + yz + zx = c(x + y + z)$

(e) $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0$, **Ans.** $x^2 + y^2 + z(x + y) = cxy$

3 Solve the following equations using Method II

(a) $y(y + z)dx + x(x - z)dy + x(x + y)dz = 0$, **Ans.** $x(y + z) = c(x + y)$

(b) $(x - y)dx - xdy + zdz = 0$, **Ans.** $x^2 - 2xy + z^2 = c$

(c) $yz^2(x^2 - yz)dx + x^2z(y^2 - xz)dy + xy^2(z^2 - xy)dz = 0$, **Ans.** $x^2z + y^2x + z^2y = cxyz$

(d) $(2z^2 - xy + y^2)zdx + (2z^2 + x^2 - xy)zdy - (x + y)(xy + z^2)dz = 0$, **Ans.** $(x + y)^2z = c(z^2 - xy)$

4 Solve the following equations using Method *III*
(a) $xz^3dx - zdy + 2ydz = 0$, **Ans.** $x^2z^2 - 2y = cz^2$
(b) $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$, **Ans.** $x^3 + y^3 = e^{2z} + ce^z$
(c) $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0$, **Ans.** $xk = x^2 + y^2 + z^2$
(d) $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$ **IAS : 1999**, **Ans.** $(x + z)y = k(y + z)$

5 Solve the following equations using Method *IV*
(a) $(y^2 + z^2 + x^2)dx - 2xydy - 2xzdz = 0$, **VU(CBCS):2018** **Ans.** $y^2 + z^2 = x^2 + xk$
(b) $yz \log xdx - zx \log zdy + xydz = 0$, **Ans.** $x \log z = cy$
(c) $zydx + (x^2y - xz)dy + (x^2y - xy)dy + (x^2z - xy)dz = 0$, **CU(H):2016** **Ans.** $x^2(y^2 + z^2 - 2c) = 2xyz$
(d) $(mz - ny)dx + (nx - lz)dy + (ly - mx)dz = 0$, **Ans.** $nx - lz = c(mz - ny)$

6 Solve $2xzdx + zdy - dz = 0$. **Ans.** $x^2 + y - \log z = k$ is the required solution where k is arbitrary constant.

7 Solve $2xzdx + zdy - dz = 0$. **Ans.** $x^2 + y - \log z = c$.

8 Solve $(2xz - yz)dx + (2yz - xz)dy - (x^2 - xy + y^2)dz = 0$. **Ans.** $x^2 - xy + y^2 = cz$.

9 Solve $(x^2 + xy + yz)dx - x(x + z)dy + x^2dz = 0$. **Ans.** $x + z = ce^{\frac{y}{x}}$.

10 Find $f(y)$ such that the total differential equation $\frac{(yz+z)}{x}dx - zdy + f(y)dz = 0$ is integrable. Hence solve it. **VU(CBCS): 2018** **Ans.** $f(y) = k(y + 1)$ and required solution is $xz^k = c(y + 1)$ where k, c are arbitrary constants.

11 Find the general solution of the equation $ydx + (z - y)dy + xdz = 0$ which is consistent with the relation $2x - y - z = 1$. **Ans.** $x^2 + xy - y^2 - y = k$ is the required solution where k is arbitrary constant.

12 Find the system of curves lying on the system of surfaces $xz = c$ and satisfying the differential equation $yzdx + z^2dy + y(z + x)dz = 0$. **Ans.** System of curves are $x = c_1y$ and $xz = c_2$ where c_1, c_2 are arbitrary constant.

13 Show that the curves satisfying the differential equation $xdx + ydy + c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}dz = 0$ that lie on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ lie also on the family of concentric spheres $x^2 + y^2 + z^2 = a^2$.

14 Find the orthogonal trajectories on the conicoid $(x + y)z = 1$ of the conics in which it which it is cut by the system of planes $x - y + z = k$, where k is parameter. **Ans.** $(x + y)z = c$ and $x + c' = z - \frac{1}{6z^3} + \frac{1}{2z}$.

15 Find $f(y)$ if $f(y)dx - zxdy - xy \log ydz = 0$ is integrable. **Ans.** $f(y) = cy$

16 Solve: $z(z - y)dx + z(z + x)dy + x(x + y)dz = 0$, **Ans.** $x + z = cz(x + y)$ where c is an arbitrary constant.

17 Solve : $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$. **Ans.** $x^3 + y^3 = e^{2z} + ce^z$ where c is an arbitrary constant.

18 Solve : $(\cos x + e^xy)dx + (e^x + e^yz)dy + e^ydz = 0$. **Ans.** $yce^x + ze^y + \sin x = c$ where c is an arbitrary constant.

19 Solve : $(z + z^2) \cos xdx - (z + z^2)dy + (1 - z^2)(y - \sin x)dz = 0$. **Ans.** $y = \sin x - cze^{-z}$ where c is an arbitrary constant.

Bibliography

[1] Birkhoff, G. and Rota, G.C. (1989). Ordinary Differential Equations (4th ed.), Wiley, New York.

[2] Bernstein, D.L., (1951). Existence Theorems in Partial Differential Equations (AM-23), Annals of Mathematics Studies, Princeton University Press.

[3] Boyce, W. E. and DiPrima, R. C. (2001). Elementary Differential Equations and Boundary Value Problems, John Wiley & Sons, Inc. New York.

[4] Chatterjee, D., (2000). Integral Calculus and Differential Equations, Tata McGraw-Hill Publishing Company Limited, New Delhi.

[5] Coddington, E. A.(1961). An Introduction to Ordinary Differential Equations, Dover Publications, Inc., New York.

[6] Debnath, L. and Bhatta, D. (2006). Integral Transforms and Their Applications, Chapman & Hall/CRC Taylor & Francis Group.

[7] Jana, D. K. and Maity, K. (2013). Advance Engineering Numerical Methods, Macmillan Publisher India Ltd.

[8] Jordan, D. W. and Smith, P. (2009). Nonlinear Ordinary Differential Equations, Oxford University Press, Oxford.

[9] Jain, M. K., Iyengar, S.R.K. and Jain, R.K. (1984). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi.

[10] Ghosh, R. K. and Maity, K. C.(2011). An Introduction to Differential Equations (9th Edition) New Central Book Agency (P) Ltd., Kolkata.

[11] Khatua, D. and Maity, K.(2016). ”Stability Analysis of a Dynamical System”, (edited by Maity, K.) in ”Bio-mathematical Modeling under Uncertain Environment”. NAROSA, New Delhi.

[12] Liao, X., Wang, L. and Yu, P. (2007). Stability of a dynamical system. Elsevier.

[13] Liu, W.M. (1994). Criterion of Hopf bifurcation without using eigenvalues. J. Math. Anal. Appl, 182 : 250.

[14] Ma, Z.,Wang, Y. and Jiang, G. (2012). Bifurcation analysis of a linear Hamiltonian system with two kinds of impulsive control. Nonlinear Dyn., 70 : 2367-2374.

[15] Maity, K. (2016). Bio-Mathematical Modelling Under Uncertain Environment(Edited Book), Narosa, New Delhi.

[16] Makinde, O.D. (2007). Solving ratio-dependent predatorprey system with constant effort harvesting using Adomain decomposition method. Appl. Math.Comput, 187 : 17-22.

[17] Mazandarani, M., Najariyan, M.,(2014). A note on ”A class of linear differential dynamical systems with fuzzy initial condition”, Fuzzy Sets and Systems, Article in press.

[18] Mondal, N.(2015). Differential Equations (Ordinary and Partial), Books and Applied (P) Ltd, Kolkata.

[19] Mondal, S. K.,(2016). ”An Overview on Stability Analysis of a Nonlinear System and its Application in Prey-Predator Model”, (edited by Maity, K.) in ”Bio-Mathematical Modelling Under Uncertain Environment”, Narosa, New Delhi.

[20] Morley, M. (1965). Categoricity in Power. Transactions of the American Mathematical Society (American Mathematical Society), 114 (2): 514-538.

[21] Mukherjee, A., Bej, N. K. (2009). Ordinary and Partial Differential Equations, Shreedhar Prakashani, Kolkata.

[22] Xie, W. C.(2010). Differential Equations for Engineers, Cambridge University Press, Cambridge.

[23] Parks, P.C. and Hahn, V.,(1992). Stability Theory, Prentics Hall, New York.

[24] Perko, L. (1991). Differential Equations and Dynamical Systems. Springer.

[25] Raisinghania, M,D. (2011). Ordinary and Partial Differential Equations, S. Chand and Company Ltd., New Delhi.

[26] Rao, K. S.(2009). Introduction to Partial Differential Equations, Phi Learning Private Limited, New Delhi.

[27] Ross, S. L.(2004). Differential Equations, John Wiley & Sons(Asia) Pre. Ltd., Singapore.

[28] Shelah, S. (1969). Stable theories. Israel J. Math., 7 (3): 187-202.

[29] Shelah, S. (1990). Classification theory and the number of nonisomorphic models. Studies in Logic and the Foundations of Mathematics (2nd ed.), Elsevier.

[30] Somasundaram, D. (2010). Ordinary Differential Equations-A first cource, Narosa Publishing House.

[31] Spiegel, M.R. (1965). Theory and Problems of Laplace Transforms, Schaum’s Outline Series, MacGraw-Hill Book Company, Inc.

[32] Weinberger, H. (1995). A First Course in Partial Differential Equations, Wiley, New York.

[33] Xiao, D., Li, W. and Han, W. (2006). Dynamics in a ratio dependent predator prey modelwith predator harvesting. J. Math.Anal. Appl., 324(1): 4-29.

Index